Lecture 1

Apolarity

1.1 Apolar schemes

1.1.1 The apolar ring of a homogeneous form

Let V be a complex vector space of dimension $n+1$. Recall from Chapter 1 that we have a natural polarity pairing

$$S^k(V) \times S^d(V^*) \to S^{d-k}(V^*), \quad (\Phi, F) \mapsto P_\Phi(F), \quad d \geq k,$$

which extends the canonical pairing $V \times V^* \to \mathbb{C}$. By choosing a basis in V and the dual basis in V^*, we view the ring $\mathrm{Sym}(V^*)$ as the polynomial algebra $\mathbb{C}[T_0, \ldots, T_n]$ and $\mathrm{Sym}(V)$ as the ring of differential operators $\mathbb{C}[\partial_0, \ldots, \partial_n]$. The polarity pairing is induced by the natural action of operators on polynomials.

Definition 1.1.1. *A homogeneous form $\Phi \in S^k(V)$ is called* apolar to a homogeneous form $F \in S^d(V^*)$ if $P_\Phi(F) = 0$.

Lemma 1.1.1. *For any $\Phi \in S^k(V), \Phi' \in S^{k'}(V)$ and $F \in S^d(V^*)$,*

$$P_{\Phi'}(P_\Phi(F)) = P_{\Phi\Phi'}(F).$$

Proof. By linearity and induction on the degree, it suffices to verify the assertions in the case when $\Phi = \partial_i$ and $\Phi' = \partial_j$. In this case the assertions are obvious. $\square$

Corollary 1.1.2. *Let $F \in S^d(V^*)$. Let $\mathrm{AP}_k(F)$ be the subspace in $S^k(V)$ spanned by apolar forms of degree k to F. Then*

$$\mathrm{AP}(F) = \bigoplus_{k=0}^{\infty} AP_k(F)$$

is a homogeneous ideal in the ring $\mathrm{Sym}(V)$.

Definition 1.1.2. *The quotient ring*

$$A_F = \mathrm{Sym}(V)/\mathrm{AP}(F)$$

is called the apolar ring of F.

The ring A_F inherits the grading of $\mathrm{Sym}(V)$. Since any polynomial $\Phi \in S^r(V)$ with $r > d$ is apolar to F, we see that A_F is killed by the ideaL $\mathfrak{m}_+^{d+1} = (\partial_0, \ldots, \partial_n)^{d+1}$. Thus A_F is an Artinian graded local algebra over $\mathbb{C}$. Since the pairing between $S^d(V)$ and $S^d(V^*)$ has values in $S^0(V^*) = \mathbb{C}$, we see that $\mathrm{AP}(F)_d$ is of codimension 1 in $S^d(V)$. Thus $(A_F)_d$ is a vector space of dimension 1 over $\mathbb{C}$ and coincides with the *socle* of A_F, i.e. the ideal of elements of A_F annulated by its maximal ideal.

Note that the latter property characterizes Gorenstein graded local Artinian rings, see [Eisenbud, Iarobbino-Kanev].

Proposition 1.1.3. *(Macaulay) The correspondence $F \mapsto A_F$ is a bijection between $\mathbb{P}(S^d(V))$ and graded Artinian quotient algebras* $\mathrm{Sym}(V)/I$ *whose socle is one-dimensional.*

Proof. We have only to show how to reconstruct $\mathbb{C}F$ from $S(V)/I$. The multiplication of d vectors in V composed with the projection to $S^d(V)/I_d$ defines a linear map $S^d(V) \to S^d(V)/I_d$. Since $(S(V)/I)_d$ is one-dimensional. Choosing a basis $(S(V)/I)_d$, we obtain a linear function on $S^d(V)$. It corresponds to an element of $S^d(V^*)$. This is our F.

□

Recall that for any closed subscheme $Z \subset \mathbb{P}^n$ is defined by a unique saturated homogeneous ideal I_Z in $\mathbb{C}[T_0, \ldots, T_n]$. Its locus of zeroes in the affine space $\mathbb{A}^{n+1}$ isomorphic to $\mathrm{Spec}(\mathbb{C}[T_0, \ldots, T_n]/I_Z)$ is the affine cone C_Z over Z.

Definition 1.1.3. *Let $F \in S^d(V^*)$. A subscheme $Z \subset \mathbb{P}(V)$ is called* apolar to F if its homogeneous ideal I_Z is contained in $\mathrm{AP}(F)$, or, equivalently, $\mathrm{Spec}(A_F)$ is a closed subscheme of the affine cone C_Z of Z.

This definition agrees with the definition of an apolar homogeneneous form Φ. A homogeneous form $\Phi \in S^k(V)$ is apolar to F if and only if the hypersurface $V(\Phi)$ is apolar to F.

Consider the natural pairing

$$(A_F)_k \times (A_F)_{d-k} \to (A_F)_d \cong \mathbb{C} \tag{1.1}$$

defined by multiplication of polynomials. The left kernel of this pairing consists of $\Phi \in S^k(V) \mod \mathrm{AP}(F)$ such that $P_{\Phi\Phi'}(F) = 0$ for all $\Phi' \in S^{d-k}(V)$. By Lemma 1.1.1, $P_{\Phi\Phi'}(F) = P_{\Phi'}(P_\Phi(F)) = 0$ for all $\Phi' \in S^{d-k}(V)$. This implies $P_\Phi(F) = 0$. Thus $\Phi \in$ AP is zero in AP(F). This shows that the pairing (1.1) is a perfect pairing. This is one of the nice features of a Gorenstein artinian algebra.

It follows that the Hilbert poynomial

$$H_{A_F}(t) = \sum_{i=0}^d \dim(A_F))it^i = a_o + \ldots + a_d t^d$$

is a reciprocal monic polynomial, i.e. $a_i = a_{d-i}, a_d = 1$. It is an important invariant of a homogenous form F.

Example 1.1.1. Let F be the dth power of a linear form $l \in V^*$. For any $\Phi \in S^k(V) = S^k((V^*)^*)$ we have

$$P_\Phi(l^d) = d(d-1)\ldots(d-k+1)l^{d-k}\Phi(l) = d!l^{[d-k]}\Phi(l),$$

where we set

$$l^{[i]} = \frac{1}{i!}l^i.$$

Here we view $\Phi \in S^d(V)$ as a homogeneous function on V^*. In coordinates, $l = \sum_{i=0}^n a_i T_i, \Phi = \Phi(\partial_0, \ldots, \partial_n)$ and $\Phi(l) = \Phi(a_0, \ldots, a_n)$. Thus we see that $AP(F)_k, k \leq d$, consists of polynomial of degree k vanishing at l. Assume for simplicity that $l = T_0$. The ideal $AP(F)$ is generated by $\partial_1, \partial_n, \partial_0^{d+1}$. The Hilbert polynomial is equal to $1 + t + \ldots + t^d$.

1.1.2 Polar polyhedra

Suppose F is equal to a sum of powers of nonzero linear forms

$$F = l_1^d + \ldots + l_s^d.$$

This implies that for any $\Phi \in S^k(V)$,

$$P_\Phi(F) = P_\Phi(\sum_{i=1}^s l_i^d) = d!\sum_{i=1}^s \Phi(l_i)l_i^{[d-k]} \tag{1.2}$$

In particular, taking $d = k$, we obtain that

$$\langle l_1^d, \ldots, l_s^d\rangle^\perp_{S^d(V)} = \{\Phi \in S^d(V) : \Phi(l_i) = 0, i = 1, \ldots, s\} = (I_Z)_d,$$

where Z is the closed subscheme of points $\{[l_1],\dots,[l_s]\} \subset \mathbb{P}(V^*)$ corresponding to the linear forms l_i.

This implies that the codimension of $\langle l_1^d,\dots,l_s^d\rangle$ in $S^d(V^*)$ is equal to the dimension of $(I_Z)_d$, hence the forms $l_1^d,\dots,l_s^d$ are linearly independent if and only if the points $[l_1],\dots,[l_s]$ impose independent conditions on the linear system of hypersurfaces of degree d in $\mathbb{P}(V)$.

Suppose $F \in \langle l_1^d,\dots,l_s^d\rangle$, then $(I_Z)_d \subset \mathrm{AP}_d(F)$. Conversely, if this is true, we have

$$F \in \mathrm{AP}_d(F)^\perp \subset (I_Z)_d^\perp = \langle l_1^d,\dots,l_s^d\rangle.$$

If we additionally assume that $(I_{Z'})_d \not\subset \mathrm{AP}_d(F)$ for any proper subset Z' of Z, we obtain, after replacing the forms $l_i's$ by proportional ones, that

$$F = l_1^d + \dots + l_s^d.$$

Definition 1.1.4. *A* polar s-polyhedron of F is a set of hyperplanes $H_i = V(l_i), i = 1,\dots,s$, in $\mathbb{P}(V^*)$ such that

$$F = l_1^d + \dots + l_s^d,$$

and, considered as points in $\mathbb{P}(V)$, the hyperplanes H_i impose independent conditions in the linear system $|\mathcal{O}_{\mathbb{P}(V)}(d)|$.

Note that this definition does not depend on the choice of linear forms defining the hyperplanes. Nor does it depend on the choice of the equation defining the hypersurface $V(F)$.

The following propositions follow from the above discussion.

Proposition 1.1.4. *Let $F \in S^d(V^*)$. Then $H_i = V(l_i) \in \mathbb{P}(V), i = 1,\dots,s$, form a polar s-polyhedron of F if and only if the following properties are satisfied*

(i) $|\mathcal{O}_{\mathbb{P}(V)}(d) - h_1 - \dots - h_s| \subset \mathrm{AP}_d(F)$*;*

(ii) $l_1^d,\dots,l_s^d$ *are linearly independent in* $S^d(V^*)$ *or*

(ii') $|\mathcal{O}_{\mathbb{P}(V)}(d) - h_1 - \dots - h_s + h_i| \not\subset \mathrm{AP}_d(F)$ *for any* $i = 1,\dots,s$.

Proposition 1.1.5. *A set $Z = \{h_1,\dots,h_s\}$ is a sth polar polyhedron of $F \in S^d(V^*)$ if and only if Z, considered as a closed subscheme of $\mathbb{P}(V)$, is apolar to F but no proper subscheme of Z is apolar to F.*

1.1.3 Generalized polar polyhedra

Proposition 1.1.5 allows one to generalize the definition of a polar polyhedron. We consider a polar polyhedron a reduced closed subscheme Z of $\mathbb{P}(V^*)$ consisting of s points. Obviously, $h^0(\mathcal{O}_Z) = \dim H^0(\mathbb{P}(V), \mathcal{O}_Z) = s$. More generally we may consider non reduced closed subschemes Z of $\mathbb{P}(V)$ of dimension 0 satisfying $h^0(\mathcal{O}_Z) = s$. The set of such subschemes is parametrized by a projective algebraic variety $\mathrm{Hilb}^s(\mathbb{P}(V))$ called the *punctual Hilbert scheme* of $\mathbb{P}(V)$ of length s.

Any $Z \in \mathrm{Hilb}^s(\mathbb{P}(V))$ defines the subspace

$$|\mathcal{O}_{\mathbb{P}(V)}(d - Z)| = \mathbb{P}(H^0(\mathbb{P}(V), \mathcal{I}_Z(d)) \subset H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) = \mathbb{P}(S^d(V)^*).$$

The exact sequence

$$0 \to H^0(\mathbb{P}(V), \mathcal{I}_Z(d)) \to H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) \to H^0(\mathbb{P}(V), \mathcal{O}_Z))$$
$$\to H^1(\mathbb{P}(V), \mathcal{I}_Z(d)) \to 0$$

shows that the dimension of the subspace

$$\langle Z\rangle_d = \mathbb{P}(H^0(\mathbb{P}(V), \mathcal{I}_Z(d))^\perp) \subset \mathbb{P}(S^d(V))$$

is equal to $h^0(\mathcal{O}_Z) - h^1(\mathcal{I}_Z(d)) - 1 = s - 1 - h^1(\mathcal{I}_Z(d))$. If $Z = Z_{red} = \{p_1, \ldots, p_s\}$, then $\langle Z\rangle_d = \langle v_d(p_1), \ldots, v_d(p_s)\rangle$, where $v_d : \mathbb{P}(V) \to \mathbb{P}(S^d(V))$ is the Veronese map. Hence $\dim\langle Z\rangle = s - 1$ if the points $v_d(p_1), \ldots, v_d(p_s)$ are linearly independent. We say that Z is *linearly d-independent* if $\dim\langle Z\rangle = s - 1$.

Definition 1.1.5. *A generalized s-polyhedron of F is a linearly d-independent subscheme* $Z \in \mathrm{Hilb}^s(\mathbb{P}(V))$ *which is apolar to F.*

Recall that Z is apolar to F if, for each $k \geq 0$,

$$H^0(\mathbb{P}^n, \mathcal{I}_Z(k)) \subset \mathrm{AP}_k(F). \tag{1.3}$$

In view of this definition a polar polyhedron is a reduced generalized polyhedron. The following is a generalization of Proposition 1.1.4.

Proposition 1.1.6. *A linear independent* $Z \in \mathrm{Hilb}^s(\mathrm{bb})V))$ *is a generalized polar polyhedron of* $F \in S^d(V^*)$ *if and only if*

$$|\mathcal{I}_Z(d)| \subset \mathrm{AP}_d(F).$$

Proof. We have to show that the inclusion in the assertion implies $\mathcal{I}_Z(d) \subset \mathrm{AP}_k(F)$ for any $k \leq d$. For any $\Phi' \in S^{d-k}(V)$ and any $\Phi \in I(Z)_k$, the product $\Phi\Phi'$ belongs to $I(Z)_d$. Thus $P_{\Phi\Phi'}(F) = 0$. By the duality, $P_\Phi(F) = 0$, i.e. $\Phi \in \mathrm{AP}_k(F)$. $\square$

Example 1.1.2. Let $Z = m_1p_1 + \ldots + m_kp_k \in \mathrm{Hilb}^s(\mathbb{P}(V))$ be the union of *fat points* p_k, i.e. at each $p_i \in Z$ the ideal $\mathcal{I}_{Z,p_i}$ is equal to the m_ith power of the maximal ideal. Obviously,

$$s = \sum_{i=1}^{k} \binom{n + m_i - 1}{m_i - 1}.$$

Then $|\mathcal{O}_{\mathbb{P}(V)}(d - Z)|$ consists of hypersurfaces of degree which have singularity at p_i of multiplicity $\geq m_i$ for each $i = 1, \ldots, k$. One can show (see [Iarrobino-Kanev], Theorem 5.3) that Z is apolar to F if and only if

$$F = l_1^{d-m_i+1}A_1 + \ldots + l_k^{d-m_k+1}A_k,$$

where $p_i = [l_i]$ and A_i is a homogeneous polynomial of degree $m_i - 1$.

Remark 1.1.1. It is not known whether the set of generalized s-polyhedra of F is a closed subset of $\mathrm{Hilb}^s(\mathbb{P}(V))$. It is known to be true for $s \leq d + 1$ since in this case $\dim I(Z)_d = t := \dim S^d(V) - s$ for all $Z \in \mathrm{Hilb}^s(\mathbb{P}(V)$ (see [IK], p.48). This defines a regular map of $\mathrm{Hilb}^s(\mathbb{P}(V))$ to the Grassmannian $G(t, S^d(V))$ and the set of generalized s-polyhedrons is equal to the preimage of a closed subset of subspaces contained in $\mathrm{AP}_d(F)$. Also we see that $h^1(\mathcal{I}_Z(d)) = 0$, hence Z is always linearly d-independent.

1.1.4 Secant varieties

The notion of a polar polyhedron has a simple geometric interpretation. Let

$$v_d : \mathbb{P}(V^*) \to \mathbb{P}(S^d(V^*)),\ l \mapsto l^d,$$

be the Veronese map. Denote by Ver_d^n its image. Then $F \in S^d(V^*)\setminus\{0\}$ represents a point $[F]$ in $\mathbb{P}(S^d(V^*))$. A set of hyperplanes $H_i = V(l_i), i = 1, \ldots, s$, represents a set of points $[l_i^d]$ in the Veronese variety Ver_d^n. It is a polar sth polyhedron of F if and only if $[F]$ belongs to the linear span $\langle [l_1^d], \ldots, [l_s^d] \rangle$, a $(s - 1)$-secant of the Veronese variety.

Recall that for any irreducible nondegenerate projective variety $X \subset \mathbb{P}^r$ of dimension n its t-*secant variety* $\mathrm{Sec}_t(X)$ is defined to be the Zariski closure of the set of points in $\mathbb{P}^r$ which lie in the linear span of dimension t of some set of $t + 1$ linear independent points in X.

Counting constants easily gives

$$\dim \mathrm{Sec}_t(X) \leq \min((n + 1)(t + 1) - 1, r).$$

The subvariety $X \subset \mathbb{P}^r$ is called *t-defective* if the inequality is strict. An example of a 1-defective variety is a Veronese surface in $\mathbb{P}^5$.

A fundamental result about secant varieties is the following lemma whose modern proof can be found, for example in [Dale], [Zak,Prop.1.10].

Lemma 1.1.7. *(A. Terracini) Let $p_1, \dots, p_{t+1}$ be general $t+1$ points in X and p be a general point in their span. Then*

$$\mathrm{PT}(\mathrm{Sec}_t(X))_p = \langle \mathrm{PT}(X)_{p_1}, \dots, \mathrm{PT}(X)_{p_{t+1}} \rangle,$$

where $\mathrm{PT}(V)_x$ denotes the embedded Zariski tangent space of a closed subvariety of a projective space at a point x.

The inclusion part

$$\langle \mathrm{PT}(X)_{p_1}, \dots, \mathrm{PT}(X)_{p_{t+1}} \rangle \subset \mathrm{PT}(\mathrm{Sec}_t(X))_p$$

is easy to prove. We assume for simplicity that $t = 1$. Then $\mathrm{Sec}_1(X)$ contains the cone $C(p_1, X)$ which is sweeped by the lines $\langle p, q\rangle, q \in X$. Therefore $\mathrm{PT}(C(p_1, X)) \subset \mathrm{PT}(\mathrm{Sec}_1(X))_p$. However, it is easy to see that $\mathrm{PT}(C(p_1, X))$ contains $\mathrm{PT}(X)_{p_1}$.

Corollary 1.1.8. $\mathrm{Sec}_t(X) \neq \mathbb{P}^r$ *if and only if for any $t+1$ general points of X there exists a hyperplane section of X singular at these points. In particular, if $r \leq (n+1)(t+1) - 1$, the variety X is t-defective if and only if for any $t+1$ general points of X there exists a hyperplane section of X singular at these points.*

Example 1.1.3. Let $X = \mathrm{Ver}_d^n \subset \mathbb{P}^{\binom{d+n}{n}-1}$ be the image of $\mathbb{P}^n$ under a Veronese map defined by homogeneous polynomials of degree d. Assume $(n+1)(t+1) \geq \binom{d+n}{n} - 1$. A hyperplane section of X is isomorphic to a hypersurface of degree d in $\mathbb{P}^n$. Thus $\mathrm{Sec}_t(\mathrm{Ver}_d^n) \neq \mathbb{P}(S^d(V^*))$ if and only if for any $t+1$ general points in $\mathbb{P}^n$ there exists a hypersurface of degree d singular at these points.

Take $n = 1$. Then $r = d$ and $r \leq (n+1)(t+1) - 1 = 2t+1$ for $t \geq (d-1)/2$. Since $t + 1 > d/2$ there are no homogeneous forms of degree d which have $t+1$ multiple roots. Thus the Veronese curve $R_d = v_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is not t-degenerate for $t \geq (d-1)/2$.

Take $n = 2$ and $d = 2$. For any two points in $\mathbb{P}^2$ there exists a conic singular at these points, namely the double line through the points. This explains why a Veronese surface V_2^2 is 1-defective.

Another example is $\mathrm{Ver}_4^2 \subset \mathbb{P}^{14}$ and $t = 4$. The expected dimension of $\mathrm{Sec}_4(X)$ is equal to 14. For any 5 points in $\mathbb{P}^2$ there exists a conic passing through these points. Taking it with multiplicity 2 we obtain a quartic which is singular at these points. This shows that Ver_4^2 is 4-defective.

The following corollary of Terracini's Lemma is called the *First main theorem on apolarity* in [Ehrenborg-Rota]. They gave an algebraic proof of this theorem without using (or probably without knowing) Terracini's Lemma.

Corollary 1.1.9. *A general form $F \in S^d(V^*)$ admits a polar sth polyhedron if and only if there exists linear forms $l_1, \ldots, l_s \in V^*$ such that for any nonzero $\Phi \in S^d(V)$ the ideal $AP(\Phi) \subset S^*(V^*)$ does not contain $\{l_1^{d-1}, \ldots, l_s^{d-1}\}$.*

Proof. A general form $F \in S^d(V^*)$ admits a polar sth polyhedron if and only if the secant variety $\mathrm{Sec}_s(\mathrm{Ver}_d^n)$ is equal to the whole space. This means that for some points $[l_1^d], \ldots, [l_s^d]$ the span of the tangent spaces at the points $\langle [l_1^d], \ldots, [l_s^d] \rangle$ is equal to the whole space. By Terracini's Lemma, this is equivalent to that the tangent spaces of the Veronese variety at the points $[l_i^d]$ are not contained in a hyperplane defined by some $\Phi \in S^d(V) = S^d(V^*)^*$. It remains to use that the tangent space of the Veronese variety at $[l_i]^d$ is equal to the projective space of all homogeneous forms of the form $l_i^{d-1}l, l \in V^*$ (see Exercise ?). Thus, for any nonzero $\Phi \in S^d(V)$, it is impossible that $P_{l_i^{d-1}l}(\Phi) = 0$ for all l and i. But $P_{l_i^{d-1}l}(\Phi) = 0$ for all l if and only if $P_{l_i^{d-1}}(\Phi) = 0$. This proves the assertion. □

The following fundamental result is due to J. Alexander and A. Hirschowitz.

Theorem 1.1.10. Ver_d^n *is t-defective if and only if*

$$(n, d, t) = (2, 2, 1), (2, 4, 4), (3, 4, 8), (4, 3, 6), (4, 4, 13).$$

In all these cases the secant variety $\mathrm{Sec}_t(\mathrm{Ver}_d^n)$ *is a hypersurface.*

For the sufficiency of the condition, only the case $(4, 3, 6)$ is not trivial. It asserts that for 7 general points in $\mathbb{P}^3$ there exists a cubic hypersurface which is singular at these points. Other cases are easy. We have seen already the first two cases. The third case follows from the existence of a quadric through 9 general points in $\mathbb{P}^3$. The square of its equation defines a quartic with 9 points. The last case is similar. For any 14 general points there exists a quadric in $\mathbb{P}^4$ containing these points.

Corollary 1.1.11. *Assume $s(n+1) \geq \binom{d+n}{n}$. Then a general homogeneous polynomial $F \in \mathbb{C}[T_0, \ldots, T_n]_d$ can be written as a sum of dth powers of s linear forms unless $(n, d, s) = (2, 2, 2), (2, 4, 5), (3, 4, 9), (4, 3, 7), (4, 4, 14)$.*

1.1.5 The Waring problems

The well-known Waring problem in number theory asks about the smallest number $s(d)$ such that each natural number can be written as a sum of $s(d)$ dth powers of natural numbers. It also asks in how many ways it can be done. Its polynomial analog asks about the smallest number $s(d,n)$ such that a general homogeneous polynomial of degree d in $n+1$ variables can be written as a sum of s dth powers of linear forms.

The Alexander-Hirschowitz Theorem completely solves this problem. We have $s(d,n)$ is equal to the smallest natural number s_0 such that $s_0(n+1) \geq \binom{n+d}{n}$ unless $(n,d) = (2,2),(2,4),(3,4),(4,3),(4,4)$, where $s(d,n) = s_0+1$.

Other versions of the Waring problem ask the following questions:

- (W1) Given a homogeneous form $F \in \mathbb{C}[T_0,\ldots,T_n]$, study the subvariety $W(F;s)^o$ of $(\mathbb{P}^n)^{(s)}$ which consists of polar s-polyhedra of F or more general the subvariety $W(F;s)$ of $\mathrm{Hilb}^s(\mathbb{P}^n)$ parametrizing generalized s-polyhedra.
- (W2) For given s find the equations of the closure $\mathrm{PS}(s,d;n)$ in $\mathbb{C}[T_0,\ldots,T_n]_d$ of the locus of homogeneous forms of degree d which can be written as a sum of s powers of linear forms.

Note that $\mathrm{PS}(s,d;n)$ is the affine cone over the secant variety $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^n)$.

In the language of secant varieties, the variety $W(F;s)^o$ is the set of linear independent sets of s points $p_1,\ldots,p_s$ in Ver_d^n such that $[F] \in \langle p_1,\ldots,p_s\rangle$. The variety $W(F,s)$ is the set of linearly independent $Z \in \mathrm{Hilb}^s(\mathbb{P}(V))$ such that $[F] \in \langle Z\rangle$. Note that we have a natural map

$$W(F,s) \to G(s, S^d(V)), Z \mapsto \langle Z\rangle,$$

where $G(s,S^d(V))$ is the Grassmannian of s-dimensional subspaces of $S^d(V)$. This map is not injective in general.

Also note that $W(F,s)^o$ embeds naturally in $\mathbb{P}(S^d(V^*))$ by assigning to $\{l_1,\ldots,l_s\}$ the product $l_1\cdots l_s$. Thus we can compactify $W(F,s)^o$ by taking its closure in $\mathbb{P}(S^d(V^*))$. In general this closure is not isomorphic to $W(F,s)$.

1.2 Catalecticant matrices

Let $F \in S^d(V^*)$. Consider the linear map

$$\mathrm{ap}_k : S^k(V) \to S^{d-k}(V^*), \quad \Phi \mapsto P_\Phi(F). \tag{1.4}$$

Its kernel is the space of forms of degree k which are apolar to F.

By the polarity duality, the dual space of $S^{d-k}(V^*)$ can be identified with $S^{d-k}(V)$. Applying Lemma 1.1.1, we obtain

$${}^t(\mathrm{ap}_F^k) = \mathrm{ap}_F^{d-k}. \tag{1.5}$$

Assume that $F = \sum_{i=1}^{s} l_i^d$ for some $l_i \in V^*$. It follows from (1.2) that

$$\mathrm{ap}_F^k(S^k(V)) \subset \langle l_1^{d-k}, \ldots, l_s^{d-k}\rangle,$$

and hence

$$\mathrm{rank}(\mathrm{ap}_F^k) \le s. \tag{1.6}$$

If we choose a basis in V and a basis in V^*, then ap_F^k is given by a matrix of size $\binom{k+n}{k} \times \binom{n+d-k}{d-k}$ whose entries are linear forms in coefficients of F.

Choose a basis $\xi_0, \ldots, \xi_n$ in V and the dual basis $T_0, \ldots, T_n$ in V^*. Consider a monomial basis in $S^k(V)$ (resp. in $S^{d-k}(V^*)$) which is lexigraphically ordered. The matrix of ap_F^k with respect to these bases is called the kth *catalecticant matrix* of F and is denoted by $\mathrm{Cat}_k(F)$. Its entries $c_{\mathbf{uv}}$ are parametrized by pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$ with $|\mathbf{u}| = d-k$ and $|\mathbf{v}| = k$. If we write

$$F = d! \sum_{|\mathbf{i}|=d} \frac{1}{\mathbf{i}!} a_{\mathbf{i}} T^{\mathbf{i}},$$

then

$$c_{\mathbf{uv}} = a_{\mathbf{u}+\mathbf{v}}.$$

This follows easily from the formula

$$\partial_0^{i_0} \cdots \partial_n^{i_n}(T_0^{j_0} \cdots T_n^{j_n}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!} & \text{if} \mathbf{j} - \mathbf{i} \ge 0 \\ 0 & \text{otherwise}. \end{cases}$$

Considering $a_{\mathbf{i}}$ as independent variables $T_{\mathbf{i}}$, we obtain the definition of a general catalecticant matrix $\mathrm{Cat}_k(d, n)$.

Example 1.2.1. Let $n = 1$. Write $F = \sum_{i=0}^{d} \binom{d}{i} a_i T_0^{d-i} T_1^i$. Then

$$\mathrm{Cat}_k(F) = \begin{pmatrix} a_0 & a_1 & \ldots & a_k \\ a_1 & a_2 & \ldots & a_{k+1} \\ \vdots & \vdots & \ldots & \vdots \\ a_{d-k} & a_{d-k+1} & \ldots & a_d \end{pmatrix}$$

A matrix of this type is called a *Hankel matrix*. It follows from (1.6) that $F \in \mathrm{PS}(s, d; 1)$ implies that all $s+1 \times s+1$ minors of $\mathrm{Cat}_k(F)$ are equal to zero. Thus we obtain that $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^1)$ is contained in the subvariety of $\mathbb{P}^d$ defined by $(s+1) \times (s+1)$-minors of the matrices

$$\mathrm{Cat}_k(d, 1) = \begin{pmatrix} T_0 & T_1 & \ldots & T_k \\ T_1 & T_2 & \ldots & T_{k+1} \\ \vdots & \vdots & \ldots & \vdots \\ T_{d-k} & T_{d-k+1} & \ldots & T_d \end{pmatrix}, \quad k = 1, \ldots, \min d-s, s$$

For example, if $s = 1$, we obtain that the Veronese curve $\mathrm{Ver}_d^1 \subset \mathbb{P}^d$ satisfies the equations $T_iT_j - T_kT_l = 0$, where $i + j = k + l$. It is well known that these equations generate the homogeneous ideal of the Veronese curve.

Assume $d = 2k$. Then the Hankel matrix is a square matrix of size $k + 1$. Its determinant vanishes if and only if F admits a nonzero apolar form of degree k. The set of such F's is a hypersurface in $\mathbb{C}[T_0, T_1]_{2k}$. It contains a Zariski open subset of forms which can be written as a sum of k powers of linear forms (see section 1.3.1).

For example, take $k = 2$. Then the equation

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0 \tag{1.7}$$

describes binary quartics

$$F = a_0T_0 + 4a_1T_0^3T_1 + 6a_2T_0^2T_1^2 + 4a_3T_0T_1^3 + a_4T_1^4$$

which lie in the Zariski closure of the locus of quartics represented in the form $(\alpha_0T_0 + \beta_0T_1)^4 + (\alpha_1T_0 + \beta_1T_1)^4$. Note that a quartic of this form has simple roots unless it has a root of multiplicity 4. Thus any binary quartic with simple roots satisfying equation (1.7) can be represented as a sum of two powers of linear forms.

The cubic hypersurface in $\mathbb{P}^4$ defined by equation (1.7) is equal to the 1-secant variety of a Veronese curve in $\mathbb{P}^4$.

Recall that each $F \in S^d(V^*)$ defines its apolar ring $A_F = \mathbb{C}[T_0, \dots, T_n]/\mathrm{AP}(F)$. Let

$$H_{A_F}(t) = \sum_{i=0}^{d} \dim(A_F)_i t^i$$

be its Hilbert polynomial. Note that

$$\dim \mathrm{AP}_i(F) = \dim \mathrm{Ker}(\mathrm{ap}_F^i) = \binom{n+k}{i} - \mathrm{rankCat}_i(F).$$

Therefore,

$$\dim(A_F)_i = \mathrm{rankCat}_i(F),$$

and

$$H_{A_F}(t) = \sum_{i=0}^{d} \mathrm{rankCat}_i(F)t^i. \tag{1.8}$$

It follows from (1.5) that

$$\mathrm{rankCat}_i(F) = \mathrm{rankCat}_{d-i}(F)$$

confirming that $H_{A_F}(t)$ is a reciprocal monic polynomial.

Suppose $d = 2k$ is even. Then the coefficient at t^k in $H_F(t)$ is equal to rankCat$_k(F)$. The matrix Cat$_k(F)$ is a square matrix of size $\binom{n+k}{k}$. One can show that for a general F, this matrix is nonsingular. A polynomial F is called *degenerate* if $\det \mathrm{Cat}_k(F) = 0$. Thus, the set of degenerate polynomials is a hypersurface given by the equation

$$\det \mathrm{Cat}_k(2k, n) = 0. \tag{1.9}$$

The polynomial in variables $T_{\mathbf{i}}, |\mathbf{i}| = d$ is called the *catalecticant determinant.*

Example 1.2.2. Let $d = 2$. It is easy to see that the catalecticant polynomial is the discriminant polynomial. Thus a quadratic form is degenerate if and only if it is degenerate in the usual sense. The Hilbert polynomial of a quadratic form F is

$$H_F(t) = 1 + rt + t^2,$$

where r is the rank of the quadratic form.

Example 1.2.3. Suppose $F = T_0^d + \ldots + T_s^d, s \leq n$. Then $T_0^i, \ldots, T_s^i$ are linearly independent for any i, and hence rankCat$_i(F) = s$ for $0 < i < d$. This shows that

$$H_F(t) = 1 + s(t + \ldots + t^{d-1}) + t^d.$$

Let $\mathcal{P}$ be the set of reciprocal monic polynomials of degree d. One can stratify the space $S^d(V^*)$ by setting, for any $P \in \mathcal{P}$,

$$S^d(V^*)_P = \{F \in S^d(V^*) : H_F = P\}.$$

If $F \in \mathrm{PS}(s, d; n)$ we know that

$$\mathrm{rankCat}_k(F) \leq h(s, d, n)_k = \min(s, \tbinom{n+k}{n}, \tbinom{n+d-k}{n}).$$

One can show that for general enough F, we have the equality (see [Iarrobono-Kanev, Lemma 1.7]). Thus there is a Zariski open subset of PS$(s, d; n)$ which belongs to the strata $S^d(V^*)_P$, where $P = \sum_{i=0}^d h(s, d, n)_i t^i$.

1.3 Examples

1.3.1 Binary forms

This is the case $n = 1$. The zero subscheme of a homogeneous form of degree d in 2 variables $F(T_0, T_1)$ is a positive divisor $D = \sum m_i p_i$ of degree d. Each such divisor is obtained in this way. Thus we can identify $\mathbb{P}(S^d(V^*))$ with $|\mathcal{O}_{\mathbb{P}(V)}(d)| \cong \mathbb{P}^d$ and also with the symmetric product $\mathbb{P}(V)^{(d)} = \mathbb{P}(V)^d / S_d$ and the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}(V))$. A generalized s-polyhedron of F is a positive divisor $Z = \sum_{i=1}^k m_i[l_i]$ of degree s in $\mathbb{P}(V^*)$ such that $[F] \in \langle Z\rangle = \mathbb{P}(H^0(\mathbb{P}(V), \mathcal{I}_Z(d))^\perp)$. Note that in our case Z is automatically linearly independent (because $H^1(\mathcal{I}_Z(d) = 0)$. Obviously, $H^0(\mathbb{P}(V), \mathcal{I}_Z(d))$ consists of polynomials of degree d which are divisible by $\Phi = \xi_1^{m_1} \cdots xi_k^{m_k}$, where $\xi_i \in \mathrm{AP}(l_i)_1$. In coordinates, if $l_i = a_iT_0 + b_iT_1$, then $\xi_i = b_i\partial_0 - a_i\partial_1$. Thus F is orthogonal to this space if and only if $P_{\Phi\Phi'}(F) = 0$ for all $\Phi' \in S^{d-s}(V)$. By the apolarity duality this means that $P_\Phi(F) = 0$, hence $\Phi \in \mathrm{AP}(F)_s$. Thus we obtain

Theorem 1.3.1. *A positive divisor $Z = V(l_1^{m_1} \cdots l_k^{m_k})$ of degree is a generalized s-polyhedron of F if and only if $\xi_1^{m_1} \cdots \xi_k^{m_k} \in \mathrm{AP}(F)_s$.*

Corollary 1.3.2. *Assume $n = 1$. Then*

$$W(F; s) = \mathbb{P}(\mathrm{AP}_s(F)).$$

Note that the kernel of the map

$$S^s(V) \to S^{d-s}(V^*), \quad \Phi \mapsto P_\Phi(F)$$

is of dimension $\geq \dim S^s(V) - \dim S^{d-s}(V^*) = s + 1 - (d - s + 1) = 2s - d$. Thus $P_\Phi(F) = 0$ for some nonzero $\Phi \in S^s(V)$, whenever $2s > d$. This shows that a F has always generilized polar sth polyhedron for $s > d/2$. If d is even, a binary form has an apolar $d/2$-form if and only $\det \mathrm{Cat}_{d/2}(F) = 0$. This is a divisor in the space of all binary d-forms.

Example 1.3.1. Take $d = 3$. Assume that F admits a polar 2-polyhedron. Then

$$F = (a_1T_0 + b_1T_1)^3 + (a_2T_0 + b_2T_1)^3.$$

It is clear that F has 3 distinct roots. Thus, if $F = (a_1T_0 + b_1T_1)^2(a_2T_0 + b_2T - 1)^2$ has a double root, it does not admit a polar 2-polyhedron. However, it admits a generalized 2-polyhedron defined by the divisor $2p$, where $p = (b_1, -a_1)$. In the secant variety interpretation, we know that any point in $\mathbb{P}(S^3(V^*))$ either lies on a unique secant or on a unique tangent line of the Veronese cubic curve. The space $\mathrm{AP}(F)_2$ is always one-dimensional. It is generated either by a binary quadric $(-b_1\xi_0 + a_1\xi_1)(-b_2\xi_0 + a_2\xi_1)$ or by $(-b_1\xi_0 + a_1\xi_1)^2$.

Thus $W(F; 2)^o$ consists of one point or empty but $W(F; 2)$ always consist of one point. This example shows that $W(F; 2) \neq \overline{W(F; 2)^o}$ in general.

1.3.2 Quadrics

It follows from Example 1.1.1 that $\mathrm{Sec}_t(Ver_2^n) \neq \mathbb{P}(S^2(V^*))$ if only if there exists a quadric with $t + 1$ singular points in general position. Since the singular locus of a quadric $V(Q)$ is a linear subspace of dimension equal to $\mathrm{corank}(Q) - 1$, we obtain that $\mathrm{Sec}_{n+1}(\mathrm{Ver}_2^n) = \mathbb{P}(S^2(V^*))$, hence any general quadratic form can be written as a sum of $n+1$ squares of linear forms $l_0, \ldots, l_n$. Of course, linear algebra gives more. Any quadratic form of rank $n+1$ can be reduced to sum of squares of the coordinate functions. Thus we may assume that $Q = T_0^2 + \ldots + T_n^2$. Suppose we also have $Q = l_0^2 + \ldots + l_n^2$. Then the linear transformation $T_i \mapsto l_i$ preserves Q and hence is an orthogonal transformation. Since polar polyhedra of Q and λQ are the same, we see that the projective orthogonal group $\mathrm{PO}(n+1)$ acts transitively on the set $W(F; n+1)^o$ of polar $(n+1)$th polyhedra of Q. The stabilizer group G of the coordinate polar polyhedron is generated by permutations of coordinates and diagonal orthogonal matrices. It is isomorphic to the semi-direct product $2^n \cdot S_{n+1}$ (the Weyl group of roots systems of type B_n, D_n), where we use the standard notation 2^k for the 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^k$. Thus we obtain

Theorem 1.3.3. *Let Q be a nondegenerate quadratic form in $n+1$ variables. Then*

$$W(Q; n+1)^o \cong \mathrm{PO}(n+1)/2^n \cdot S_{n+1}.$$

The dimension of $W(Q; n+1)^o$ is equal to $\frac{1}{2}n(n+1)$.

Example 1.3.2. Take $n = 1$. Using the Veronese map $v_2 : \mathbb{P}^1 \to \mathbb{P}^2$, we consider a nonsingular quadric $V(Q)$ as a point p in $\mathbb{P}^2$ not lying on the conic $C = V(T_0T_2 - T_1^2)$. A polar 2-gon of Q is a pair of distinct points p_1, p_2 on C such that $p \in \langle p_1, p_2 \rangle$. It can be identified with the pencil of lines through p with the two tangent lines to C deleted. Thus $W(Q, 2)^o = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. There are two generalized 2-gons $2p_0$ and $2p_\infty$ defined by the tangent lines. Each of them gives the representation of Q as $l_1 l_2$, where $V(l_i)$ are the tangents. We have $W(F; 2) = \overline{W(F; 2)^o} \cong \mathbb{P}^1$.

It is an interesting question to define a good compactification of this space similar to the one we found in Chapter 2 in the case $n = 2$.

Let $Q \in S^2(V^*)$ be a non-degenerate quadratic form. Each $\{l_1, \ldots, l_{n+1}\} \in W(Q; n+1)^o$ defines a $(n+1)$-dimensional subspace in $E = S^2(V^*)/\langle Q \rangle \cong \mathbb{C}^{\binom{n+2}{2}-1}$ equal to $\langle l_1^2, \ldots, l_{n+1}^2 \rangle \mod \langle Q \rangle$. This defines a map

$$f : W(Q; n+1)^o \to G(n, E). \tag{1.10}$$

This map is injective. In fact, suppose $\langle l_1^2, \ldots, l_{n+1}^2 \rangle = \langle m_1^2, \ldots, m_{n+1}^2 \rangle \mod \langle F \rangle$. Then

$$l_1^2 = a_1 m_1^2 + \ldots + a_{n+1} m_{n+1}^2 + \lambda F$$

for some scalars $a_1, \ldots, a_{n+1}, \lambda$. Since F is a linear combination of m_i^2, we may assume that $\lambda = 0$. But l_1^2, being of rank 1, cannot be equal to a sum of k linear independent squares of linear forms unless $k = 1$. Thus we may assume that $l_i^2 = a_i m_i^2$ for some i. This immediately implies F is a sum of n squares on linear forms contradicting the assumption that F is of rank $n+1$.

Now consider the dual quadratic form $Q^* \in S^2(V)$ of Q considered as a linear function on $S^2(V^*)$. For any two quadratic forms $A, B \in S^2(V^*)$ we can consider the matrix

$$\begin{pmatrix} A_0 & A_1 & \ldots & A_n \\ B_0 & B_1 & \ldots & B_n \end{pmatrix}, \tag{1.11}$$

where A_i, B_i are partial derivatives in the variable T_i. Let $\Delta_{ij}, 0 \le i < j \le n$, be the 2×2 minors of the matrix. The function $\sigma_{ij} : (A, B) \mapsto Q^*(\Delta_j)$ is a skew-symmetric bilinear form on the space $S^2(V^*)$. Of course, to define the partial derivatives we need to choose a basis in V. However, a change of a basis multiplies the matrix (1.11) by an invertible constant matrix, and hence σ_{ij} is unchanged up to a nozero scalar factor. Thus, to be more precise,

$$\sigma_{ij} \in \mathbb{P}(\Lambda^2(S^2(V^*)^*).$$

I claim that $\sigma_{ij}(Q, A) = 0$ for any $A \in S^2(V^*)$. In fact, choose a basis in V such that $Q = \sum_{i=0}^n T_i^2$. Then $Q^* = \sum_{i=0}^n \partial_i^2$. By linearity, we may assume that $A = T_iT_j,\ i \le j$.

$$\sigma_{kl}(Q, A) = \sum_{i=0}^n \partial_i^2((2T_k\partial_l(T_iT_j) - 2T_l\partial_k(T_iT_j)).$$

If $k \ne i, l \ne i, j$ we get zero. If $k = i, l \ne j$, we get

$$\sigma_{kl}(Q, A) = \sum_{i=0}^n \partial_i^2(-2T_lT_j) = 0.$$

Finally, if $k = i, l = j$, we get

$$\sigma_{kl}(Q, A) = \sum_{i=0}^n \partial_i^2(2T_i^2 - 2T_j^2) = 0.$$

Since Q belongs to the kernel of the bilinear form σ_{ij} we can consider each function σ_{ij} as a skew Recall that the Grassmann variety $G(n, E)$ carries the natural rank n vector bundle $\mathcal{S}$, the *tautological bundle*. Its fibre over a point $L \in G(n, E)$ is equal to L. It is a subbundle of the trivial bundle $E_{G(3,E)}$ associated to the vector space E. We have a natural exact sequence

$$0 \to \mathcal{S} \to E_{G(3,E)} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the *universal quotient bundle*, whose fibre over L is equal to E/L. We can consider each element ω of $\Lambda^2(E^*)$ as a section of the trivial bundle $\Lambda^2(E^*_{G(3,E)})$. Restricting ω to the subbundle $\mathcal{S}$, we get a section of the vector bundle $\Lambda^2(\mathcal{S}^*)$. Thus we can view our functions σ_{ij} as sections of $\Lambda^2(\mathcal{S}^*)$ defined up to a constant, i.e. elements of the projective space $\mathbb{P}(H^0(G(n,E),\Lambda^2(\mathcal{S}^*)))$.

Lemma 1.3.4. *The image of the map* (1.10) *is contained in the set of common zeros of the sections σ_{ij}.*

Proof. We have to show that σ_{ij} is identical to zero on each subspace $\langle l_1^2,\ldots,l_{n+1}^2\rangle$, containing Q. Again, without loss of generality, we may assume that $Q=\sum T_s^2, Q^*=\sum_t \partial_s^2$. By linearity, it suffices to show that $\sigma_{kl}(l_i^2,l_j^2)=0$ for all $0\le k<l\le n+1, 1\le i<j\le n+1$. We have

$$\sigma_{kl}(l_i^2,l_j^2)=\sum_{s=0}^n \partial_s^2\big(\partial_k(l_i^2)\partial_l(l_j^2)-\partial_k(l_j^2)\partial_l(l_i^2)\big)=$$

$$2\sum_{s=0}^n \partial_s^2(l_il_j)\big(\partial_k(l_i)\partial_l(l_j)-\partial_k(l_j)\partial_l(l_i)\big).$$

Let $l_i=\sum_{s=0}^n a_sT_s,\ l_j=\sum_{s=0}^n b_sT_s$. Since $\sum_{s=0}^n l_s^2=\sum_{s=0}^n T_s^2$, we easily see, that the coefficients of the linear forms l_s, considered as vectors in $\mathbb{C}^{n+1}$, are orthogonal with respect to the dot-product. Therefore

$$\sum_{s=0}^n \partial_s^2(l_il_j)=\sum_{s=0}^n a_sb_s=0.$$

This proves the assertion. □

The next result has been proven already, by different method, in Chapter 2, Part I.

Corollary 1.3.5. *Assume $n=2$. Then the image of $W(Q;3)^o$ in $G(2,E)$, embedded in the Plücker space $\mathbb{P}(\Lambda^2(E))$ is an open Zariski subset of the intersection X of $G(2,E)$ with a linear space of codimension 3.*

Proof. We have $\dim E=5$, so $G(2,E)\cong G(2,5)$ is of dimension 6. Hyperplanes in the Plücker space are elements of the space $\mathbb{P}(\Lambda^2(E)^*)$. Note that the functions σ_{ij} are linearly independent. In fact, assume $Q=\sum T_i^2$. if we take $A=T_0T_1+T_2^2, B=-T_0^2+T_1^2+T_2^2$, we see that $\sigma_{01}(A,B)\neq 0,\sigma_{12}(A,B)=0,\sigma_{02}(A,B)=0$. Thus a linear dependence between the functions σ_{ij} implies the linear dependence between two of the functions. It is easy to see that no two functions are proportional. So our 3 functions $\sigma_{ij}, 0 \leq i < j \leq 2$ span a codimension 3 subspace L in the Plücker space. The line bundle $\mathcal{O}_{G(2,E)}(1)$ is equal to $\Lambda^2(\mathcal{S}^*)$. By the previous lemma, the image of $W(Q;3)$ is contained in the intersection $G(2,E) \cap L$. This is a 3-dimensional subvariety of $G(2,E)$, and hence contains $W(Q;3)^o$ as an open Zariski subset. □

In Chapter 2 we proved also that X is a smooth Fano 3-fold of degree 5. I do not know how to prove the smoothness of X using the present method. Note that in Chapter 2, we identified each point of X with a closed 0-dimensional subscheme Z of the Veronese surface Ver_2^2 such that $[Q] \in \langle Z \rangle$. Using our interpretation of generalized polar polyhedra, we see that $X = W(Q;3)$.

If $n > 2$, the vector bundle $\Lambda^2(\mathcal{S}^*)$ is of rank $r = \binom{n}{2} > 1$. The zero locus of its nonzero section is of expected codimension equal to r. We have $\binom{n+1}{2}$ sections σ_{ij} and $\dim G(n,E) = n(\binom{n+2}{2} - n - 1)$. For example, when $n = 3$, we have 6 sections σ_{ij} each vanishing on a codimension 3 subvariety of 18-dimensional Grassmannian $G(3,9)$. So there must be some dependence between the functions σ_{ij}.

Remark 1.3.1. One can also consider the varieties $W(Q;s)$ for $s > n+1$. For example, we have

$$\begin{aligned} T_0^2 &= \tfrac{1}{2}(T_0 + 2T_1)^2 + \tfrac{1}{2}(T_0 - 2T_1)^2 - 4T_1^2, \\ T_0^2 &= \tfrac{1}{2}(T_0 + T_1)^2 + \tfrac{1}{2}(T_0 - T_1)^2 - \tfrac{1}{2}(T_2 + T_1)^2 + \tfrac{1}{2}(T_2 - T_1)^2. \end{aligned}$$

This shows that $W(Q;n+3), W(Q;n+4)$ are not empty for any nondegenerate quadric Q. I do not know anything about their structure.

1.3.3 Cubic forms

We will start with cubic forms in 3 variables. Since for any three general points in $\mathbb{P}^2$ there exists a plane cubic singular at these points (the union of three lines), a general ternary cubic form does not admit polar triangles. Of course this is easy to see by counting constants.

A plane cubic curve projectively isomorphic to the cubic $V(T_0^3 + T_1^3 + T_2^3)$ will be called a *Fermat cubic*. Obviously such a curve admits a non-degenerate polar 3-polyhedron.

Theorem 1.3.6. *A plane cubic admits a polar 3-polyhedron if and only if either it is a Fermat cubic member or it is equal to the union of three distinct concurrent lines.*

Proof. Suppose $F = l_1^3 + l_2^3 + l_3^3$. Without loss of generality we may assume that l_1^3 is not proportional to l_2^3. Thus, after coordinate change $F = T_0^3 + T_1^3 + l^3$. If $l(T_0, T_1, T_2)$ does not depend on T_2, the curve $V(F)$ is the union of three distinct concurrent lines. Otherwise we can change coordinates to assume that $l = T_2$ and get a Fermat cubic. □

Remark 1.3.2. If F is a Fermat cubic, then its polar 3-polyhedron is unique. Its sides are the three first polars of F which are double lines.

By counting constants, we see that a general cubic admits polar quadrangles. We call a polar quadrangle $\{H_1, \ldots, H_4\}$ *nondegenerate* if it is defined by 4 points in $\mathbb{P}(V)$ no three of which are collinear. It is clear that a polar quadrangle is non-degenerate if and only if the linear system of conics in $\mathbb{P}(V)$ through the points $[l_1], \ldots, [l_4]$ is an irreducible pencil (i.e. a linear system of dimension 1 whose general member is irreducible). This allows us to define a *degenerate generalized 4-polyhedron* of F as a generalized polyhedron Z of F such that $|\mathcal{I}_Z(2)|$ is an irreducible pencil.

Lemma 1.3.7. *F admits a degenerate polar 4-polyhedron if and only if $V(F)$ is one of the following curves:*

(i) a Fermat cubic;

(ii) a cuspidal cubic;

(ii) the union of three concurrent lines (not necessary distinct);

Proof. We have

$$T_0^3 + T_1^3 + T_2^3 = \frac{1}{3}(T_0 + T_1)^3 + \frac{1}{3}(T_0 + aT_1)^3 + \frac{1}{3}(T_0 + a^2T_1)^3 + T_2^3,$$

where $a = e^{2\pi i/3}$.

We have

$$T_0T_1(9T_0 + 15T_1) = (T_0 + T_1)^3 + (T_0 + 2T_1)^3 - 2T_0^3 - 5T_1^3.$$

Since the union of three distinct concurrent lines $V(F)$ is projectively equivalent to $V(T_0T_1(9T_0 + 15T_1)$, we see that F admits a degenerate quadrangle.

We also have

$$T_0T_1^2 = (2T_0 + T_1)^3 + (T_0 - 4T_1)^3 - 9T_0^3 + 15T_1^3,$$

$$(2 + c^3)T_0^3 = (T_0 + aT_1)^3 + (T_0 + bT_1)^3 - (cT_0 + dT_1)^3 - (a^3 + b^3 - d^3)T_1^3,$$

where $a^3+b^3=c^2d, a+b=cd^2, c^3+2\neq 0$. This shows that case (iii) occurs.

All cuspidal cubics are projectively equivalently. So it is enough to demonstrate a degenerate 4-polyhedron for $V(T_0^3+6T_1^2T_2)$. We have

$$T_0^3+6T_1^2T_2=(T_1+T_2)^3+(T_2-T_1)^3-2T_2^3+T_0^3.$$

Now let us prove the converse. Suppose

$$F=l_1^3+l_2^3+l_3^3+l_4^3,$$

where l_1,l_2,l_3 vanish at a common point a. Let ξ be a linear form on V corresponding to a. We have

$$\frac{1}{3}P_\xi(F)=l_1^2\xi(l_1)+l_2^2\xi(l_2)+l_3^2\xi(l_3)+l_4^2\xi(l_4)=l_4^2\xi(l_4).$$

This shows that the first polar $P_a(F)=P_\xi(F)$ is either the whole $\mathbb{P}^2$ or a double line $V(l_4^2)$. In the first case F is the union of three concurrent lines. Assume the second case happens. We can choose coordinates such that $a=(1,0,0)$ and $l=V(T_0)$. Write

$$F=L_0T_0^3+T_0^2L_1+T_0L_2+L_3,$$

where L_i are homogeneous forms of degree i in variables T_1,T_2. Then $P_a(F)=\partial_0(F)=3T_0^2L_0+2T_0L_1+L_2$. This can be proportional to T_0^2 only if $L_1=L_2=0$. Thus $V(F)=V(T_0^3+L_3(T_1,T_2))$. If L_3 does not multiple linear factors, we can choose coordinates such that $L_3=T_1^3+T_2^3$, and get the cubic. If L_3 has a linear factor with multiplicity 2, we reduce L_3 to the form $T_1^2T_2$. This is the case of a cuspidal cubic. Finally, if L_3 is a cube of a linear form, we reduce the later to the form T_1^3 and get three concurrent lines.

□

Remark 1.3.3. The set of Fermat cubics is a hypersurface in the space $\mathbb{P}(S^3(V^*)$ isomorphic to the homogeneous space $\mathrm{PSL}(3)/3^2.S_3$. Its closure in $\mathbb{P}(S^3(V^*))$ consists of curves listed in the assertion of the previous lemma and also reducible cubics equal to the unions of irreducible conics with its tangent line.

Lemma 1.3.8. *The following properties equivalent*

(i) $\mathrm{AP}_1(F)\neq\{0\}$*;*

(ii) $\dim \mathrm{AP}_2(F)>2$*;*

(iii) $V(F)$ *is equal to the union of three concurrent lines.*

Proof. By the duality $(A_F)_1 \times (A_F)_2 \to (A_F)_3 \cong \mathbb{C}$, we have

$$\dim(A_F)_1 = 3 - \dim AP_1(F) = \dim(A_F)_2 = 6 - \dim \mathrm{AP}_2(F).$$

Thus $\dim \mathrm{AP}_2(F) = 3 + \dim AP_1(F)$. This proves the equivalence of (i) and (ii). By definition, $\mathrm{AP}(F)_1 \neq \{0\}$ if and only if $P_\Phi(F) = 0$ for some nonzero linear operator $\sum a_i \partial_i$. After a linear change of variables, we may assume that $\Phi = \partial_0$, and then $\partial_0(F) = 0$ if and only if F does not depend on T_0, i.e. $V(F)$ is the union of three concurrent lines. $\square$

Lemma 1.3.9. *Let Z be a nondegenerate generalized 4-polyhedron of F. Then $|\mathcal{I}_Z(2)|$ is a pencil contained in $|\mathrm{AP}_2(F)|$. Conversely, let Z be a 0-dimensional cycle of degree 4 in $\mathbb{P}(V)$. Assume that $|\mathcal{I}_Z(2)|$ is a pencil without fixed part contained in $|\mathrm{AP}_2(F)|$. Then Z is a nondegenerate generalized 4-polyhedron of F.*

Proof. The first assertion follows from the definition of non-degeneracy and Proposition 1.1.6. Let us prove the converse. Let $V(\lambda Q_1 + \mu Q_2)$ be the pencil of conics $|\mathcal{I}_Z(2)|$. Since $\mathrm{AP}(F)$ is an ideal, the linear system E of cubics of the form $V(Q_1L_1 + Q_2L_2)$, where L_1, L_2 are linear forms, is contained in $\mathbb{P}(\mathrm{AP}_3(F))$. Obviously it is contained in $|\mathcal{I}_Z(3)|$. Since $|\mathcal{I}_Z(2)|$ has no fixed part we may choose Q_1 and Q_2 with no common factors. Then the map $V^* \oplus V^* \to \mathcal{I}_Z(3)$ defined by $(L_1, L_2) \to Q_1L_1 + Q_2L_2$ is injective hence $\dim E = 5$. Assume $\dim |\mathcal{I}_Z(3)| \geq 6$. Choose 3 points in general position on an irreducible member C of $|\mathcal{I}_Z(2)|$ and 3 non-collinear points outside C. Then find a cubic K from $|\mathcal{I}_Z(3)|$ which passes through these points. Then K intersects C with total multiplicity $4 + 3 = 7$, hence contains C. The other component of K must be a line passing through 3 non-collinear points which is absurd. So, $\dim |\mathcal{I}_Z(3)| = 5$ and we have $E = |\mathcal{I}_Z(3)|$. Thus $|\mathcal{I}_Z(3)| \subset \mathbb{P}(\mathrm{AP}_3(F))$ and, by Proposition 1.1.6, Z is a generalized 4-polyhedron of F.

$\square$

Corollary 1.3.10. *Suppose F is not the union of three concurrent lines. The subset of $W(F, 4)$ consisting of nondegenerated generalized 4-polyhedrons is isomorphic to an open subset of the plane $|\mathrm{AP}_2(F)|^*$ consisting of pencils with no fixed part.*

Example 1.3.3. Let $V(F)$ be the union of an irreducible conic and its tangent line. After a linear change of variables we may assume that $F = T_0(T_0T_1 + T_2^2)$. It is easy to check that $\mathrm{AP}_2(F)$ is spanned by $\partial_1^2, \partial_1\partial_2, \partial_2^2 - \partial_0\partial_1$. It follows from Lemma 1.3.7 that F does not admit degenerate polar 4-polyhedra. Thus any polar 4-polyhedra of F is the base locus of an irreducible pencil in $|\mathrm{AP}_2(F)|$. However, it is easy to see that all nonsingular conics in $|\mathrm{AP}_2(F)|$ are tangent at the point $(0,1,0)$. Thus no pencil has 4 distinct base points. This shows that

$$W(F;4)^o = \emptyset.$$

Of course, $W(F,4) \neq \emptyset$. Any irreducible pencil in $|\mathrm{AP}_2(F)|$ defines a generalized 4-polyhedron. It is easy to see that the only reducible pencil is $V(\lambda\partial_1^2 + \mu\partial_1\partial_2)$. Thus $W(F,4)$ contains a subvariety isomorphic to a complement of one point in $\mathbb{P}^2 = |\mathrm{AP}_2(F)|^*$. To compactify it by $\mathbb{P}^2$ we need to find one more generalized 4-polyhedron. Consider the subscheme Z of degree 4 concentrated at the point $(1,0,0)$ with ideal at this point generated by (x^2, xy, y^3), where we use inhomogeneous coordinates $x = \partial_1/\partial_0, y = \partial_2/\partial_0$. The linear system $|\mathcal{I}_Z(3)|$ is of dimension 5 and consists of cubics of the form $V(\partial_0\partial_1(a\partial_1 + b\partial_2) + G_3(\partial_1,\partial_2))$. Thus Z is linearly 3-independent. One easily computes $\mathrm{AP}_3(F)$. It is generated by all monomials except $\partial^2\partial_1$ and $\partial_0\partial_2^2$ and also the polynomial $\partial_0\partial_2^2 - \partial^2\partial_1$. We see that $|\mathcal{I}_Z(3)| \subset \mathbb{P}(\mathrm{AP}_3(F)$. Thus Z is a generalized 4-polyhedron of F. It is nondegenerate since $|\mathcal{I}_Z(2)|$ is the pencil $V(\lambda\partial_1^2 + \mu\partial_1\partial_2)$. So, we see that $W(F,4)$ is isomorphic to the plane $|\mathrm{AP}_2(F)|^*$.

Example 1.3.4. Let $V(F)$ be an irreducible nodal cubic. Without loss of generality we may assume that $F = T_2^2T_0 + T_1^3 + T_1^2T_0$. The space of apolar quadratic forms is spanned by $\partial_0^2, \partial_1\partial_2, \partial_2^2 - \partial_1^2$. The net $|\mathrm{AP}_2(F)|$ is base-point-free. It is easy to see that its discriminant curve is the union of three distinct non-concurrent lines. Each line defines a pencil with singular general member but without fixed part. So, $W(F,4) = |\mathrm{AP}_2(F)|^*$.

Example 1.3.5. Let $V(F)$ be the union of an irreducible conic and a line which intersects the conic transversally. Without loss of generality we may assume that $F = T_0(T_0^2 + T_1T_2)$. The space of apolar quadratic forms is spanned by $\partial_1^2, \partial_2^2, 6\partial_1\partial_2 - \partial_0^2$. The net $|\mathrm{AP}_2(F)|$ is base-point-free. It is easy to see that its discriminant curve is the union of a conic and a line intersecting the conic transversally. The line defines a pencil with singular general member but without fixed part. So, $W(F,4) = |\mathrm{AP}_2(F)|^*$.

Example 1.3.6. Let $V(F)$ be a cuspidal cubic. Without loss of generality we may assume that $F = T_1^2T_0 + T_2^3$. The space of apolar quadratic forms is spanned by $\partial_0^2, \partial_0\partial_2, \partial_2\partial_1$. The net $|\mathrm{AP}_2(F)|$ has 2 base points $(0,1,0)$ and $(0,0,1)$. The point $(0,0,1)$ is a simple base-point. The point $(0,1,0)$ is of multiplicity 2 with the ideal locally defined by (x^2, y). Thus base-point scheme of any irreducible pencil is not reduced. There are no polar 4-polyhedra defined by the base-locus of a pencil of conics in $|\mathrm{AP}_2(F)|$. The discriminant curve is the inion of two lines, each defining a pencil with a fixed line. So $|\mathrm{AP}_2(F)|^*$ minus 2 points parametrizes generalized polar 4-polyhedra. We know that $V(F)$ admits degenerate polar 4-polyhedra. Thus $W(F,4)^o$ is not empty and consists of degenerate polar 4-polyhedra.

Example 1.3.7. Let $V(F)$ be a nonsingular cubic curve. We know that its equation can be reduced to a Hesse form $V(T_0^3+T_1^3+T_2^3+6aT_0T_1T_2)$, where $1+8a^3 \neq 0$. The space of apolar quadratic forms is spanned by $a\partial_0\partial_1-\partial_2^2, a\partial_1\partial_2-\partial_0^2, a\partial_0\partial_2^2-\partial_1^2$. The curve $V(F)$ is Fermat if and only if $a(a^3-1)=0$. In this case the net has 3 ordinary base points and the discriminant curve is the union of 3 non-concurrent lines. The net has 3 pencils with fixed part defined by these lines. Thus the set of nondegenerate generalized polyhedrons is equal to the complement of 3 points in $|\mathrm{AP}_2(F)|^*$. We know that a Fermat cubic admits degenerate polar 4-polyhedra.

Suppose $V(F)$ is not a Fermat cubic. Then the net $|\mathrm{AP}_2(F)|$ is base-point-free. Its discriminant curve is a nonsingular cubic. All pencils are irreducible. There are no degenerate generalized polygons. So, $W(F,4)=|\mathrm{AP}_2(F)|^*$.

Example 1.3.8. Assume that $V(F)=V(T_0T_1T_2)$ is the union of 3 non-concurrent lines. Then $\mathrm{AP}_2(F)$ is spanned by $\partial_0^2,\partial_1^2,\partial_2^2$. The net $|\mathrm{AP}_2(F)|$ is base-point-free. The discriminant curve is the union of three non-concurrent lines representing pencils without fixed point but with singular general member. Thus $W(F,4)=|\mathrm{AP}_2(F)|^*$.

It follows from the previous examples that $\mathrm{AP}_2(F)|$ is base-point-free net of conics if and only if F does not belong to the closure of the orbit of Fermat cubics.

Theorem 1.3.11. *Assume that F does not belong to the closure of the orbit of Fermat cubics. Then $|\mathrm{AP}_2(F)|$ is base-point-free net of conics and*

$$W(F,4)\cong|\mathrm{AP}_2(F)|^*\cong\mathbb{P}^2.$$

The variety $W(F,4)^o$ is isomorphic to the open subset of $|\mathrm{AP}_2(F)|^*$ whose complement is the curve B of pencils with non-reduced base-locus. The curve B is a plane sextic with 9 cusps if $V(F)$ is a nonsingular curve, the union of three non-concurrent lines if $V(F)$ is an irreducible nodal curve or the union of three lines, and the union of a conic and its two tangent lines if $V(F)$ is the union of a conic and a line.

Proof. The first assertion follows from the Examples 1.3.4-1.3.8. Since the linear system of conics $|\mathrm{AP}_2(F)|$ is base-point-free, it defines a regular map

$$\phi:\mathbb{P}(V)\to|\mathrm{AP}_2(F)|^*.$$

The pre-image of a line is a conic from $\mathcal{N}$. The lines through a point q in $|\mathrm{AP}_2(F)|^*$ define a pencil with base locus $\phi^{-1}(q)$. Thus pencils with non-reduced locus are parametrized by the branch curve B of the map ϕ.

If $V(F)$ is a nonsingular cubic, we know from Example 1.3.7 that the discriminant curve Δ is a nonsingular cubic. A line in $|\mathrm{AP}_2(F)|$ defines a pencil of conics. Its singular members are the intersection points of the line and Δ. It is easy to see that the pencil has exactly 3 singular members if and only its base-point locus consists of 4 distinct points. Thus the curve B is the dual curve of Δ. By the duality, B is dual of Δ. We know from Part 1 that it is a sextic with 9 cusps.

If $V(F)$ is an irreducible nodal curve, we know from Example 1.3.4 that Δ is the union of three non-concurrent lines. The locus of lines intersecting Δ not transversally is the union of three pencils of lines. As above B must be the union of three non-concurrent lines.

If $V(F)$ is the union of a conic and and a line, we know from Example 1.3.5 that Δ is the union of a conic and and a line intersecting the conic transversally. Obviously B must contain an irreducible component dual to the conic. Other irreducible components must be two tangent lines to the conic.

Finally if $V(F)$ is the union of three lines, the map ϕ is given by $(t_0, t_1, t_2) \to (t_0^2, t_1^2, t_2^2)$ and as is easy to see its branch locus is the union of the coordinate lines. $\square$

Let $C \subset \mathbb{P}(S^3(V^*)) \cong \mathbb{P}^9$ be the locus of three concurrent lines. For each $V(F) \in \mathbb{P}(S^3(V^*)) \setminus C$, the space $\mathrm{AP}_2(F)$ is 3-dimensional. This defines a regular map $a : \mathbb{P}(S^3(V^*)) \setminus C \to G(3, S^2(V))$. Both the varieties are 9-dimensional. Fix a 3-dimensional subspace L of $S^2(V)$ and consider the linear map

$$\tilde{a} : S^3(V^*) \to \mathrm{Hom}(L, V^*), \quad a(F)(\Phi) = P_\Phi(F).$$

Its kernel consists of cubic forms F such that $L \subset \mathrm{AP}_2(F)$. Note that the map $\tilde{a}$ is a linear map from a 10-dimensional space to a 9-dimensional space. One expects that its kernel is 1-dimensional. This shows that, for a general point $L \in G(3, S^2(V))$ the pre-image a^{-1} is a one-point. Thus the map a is birational.